Department of Pure & Applied Mathematics

Model Answer

Paper: Differential Geometry-I Paper Code:**AU-6235**

1(i). Osculating sphere and osculating plane intersect in a circle called osculating circle.

(ii) An involute of a curve C is another curve $\bar{C}$ which lies on the tangent surface of C and cuts all the tangents of C orthogonally. Equation of involute of a curve is given by

$$\boxed{x^i = x^i + (c-s)\,t^i}$$

(iii) Weingarten Equation: The equation

$$N^i_{;\alpha} = -d_{\alpha r}\, g^{r\beta}\, x^i_{;\beta}$$

is called Weingarten equation.

(iv) Equation of the curve is $\bar{r} = \bar{r}(t)$

$$\bar{r}' = \bar{t}^i \quad \text{——— (I)}$$

$$\bar{r}'' = K p^i \quad \text{——— (II)}$$

$$\bar{r}'.\bar{r}'' = K\bar{t}.p^i$$

$$= 0$$

$$\bar{r}'.\bar{r}'' = 0 \quad \text{——— (a)}$$

Now $\bar{r}''' = K' p^i + K(\tau b^i - K t^i)$ — (IV)

From (I) and (IV), we have

$$\bar{r}'.\bar{r}''' = -K^2 \quad \text{——— (b).}$$

(v) Spherical indicatrix of binormal: The locus of a point whose position vector is equal to the unit binormal at any point of a curve is called the spherical indicatrix of the binormal.

(vi) Normal Curvature: The normal curvature of a surface in the direction of a curve is denoted by K_n and defined as

$$K_n \stackrel{def}{=\!=} \frac{d_{\alpha\beta}\, du^\alpha\, du^\beta}{g_{\alpha\beta}\, du^\alpha\, du^\beta}.$$

(vii) Equation of straight line is given by

$$x^i = a^i s + b^i \quad \text{------} \quad (I)$$

differentiating w.r.t. s we have

$$x'^i = t^i = a^i \quad \text{------} \quad (II)$$

Again differentiating w.r.t. s we have

$$K p^i = 0$$

$$\Rightarrow \quad K = 0, \quad p^i \neq 0$$

Hence for straight line $K = 0$.

(viii). Curvature: The curvature at a point P of a given curve is the arc rate of rotation of tangent.

Torsion: Torsion at a point P of a curve is the arc rate of the change in the direction of binormal at P.

(ix) Rectifying Plane: The plane containing tangent t^i and binormal b^i is called rectifying plane. Equation of rectifying plane is given by

$$(x^i - x^i) \cdot p^i = 0.$$

(x). The surface generated by moving straight line is called ruled surfaces. Examples - Cone, cylinder.

2. Let the equation of osculating sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{--- (1)}$$

For the intersection of sphere and curve, we have

$$\phi(t) \equiv (2t+1)^2 + (3t^2+2)^2 + (4t^3+3)^2 + 2u(2t+1) + 2v(3t^2+2) + 2(4t^3+3)w + d = 0$$

For the point $(3, 5, 7)$, $t = 1$

Now find $\phi'(t)$, $\phi''(t)$, $\phi'''(t)$ at $t = 1$

and solving for u, v, w, and d. we get

$$u = -\frac{282}{3}, \quad v = \frac{379}{2}, \quad w = -\frac{285}{6}$$

$$d = -\frac{352}{3}$$

Putting these values in (1), we get required osculating sphere.

3. In this cas $\bar{\sigma}_1 = p^i$ ——— (I)

Now differentiate this equation w.r.t. s_1 and calculate $\frac{ds_1}{ds_1}$ i.e

$$\left(\frac{ds_1}{ds}\right)^2 = \gamma^2 + K^2 \quad \text{——— (II)}$$

Again differentiating and using (II), we can calculate curtuature and torsion given by

$$K_1 = 1 + \frac{(K\gamma' - K'\gamma)^2}{(K^2+\gamma^2)^3}$$

$$\gamma_1^2 (K^2+\gamma^2)^6 + \gamma^2 (K\gamma' - K'\gamma)^4 = K_1^4 (K^2+\gamma^2)^6 K_1'^2.$$

4. Given that

$$2z = ax^2 + 2hxy + by^2$$

$$\Rightarrow \quad X = \left(x, y, \frac{ax^2 + 2hxy + by^2}{2}\right)$$

$$X_1 = (1, 0, ax + hy)$$

$$X_2 = (0, 1, by + hx)$$

$$X_{11} = (0, 0, a)$$

$$X_{12} = (0, 0, h)$$

$$X_{22} = (0, 0, b)$$

Calculate fundamental magnitudes & normal by using

$$g_{\alpha\beta} = X_\alpha \cdot X_\beta, \qquad d_{\alpha\beta} = X_{\alpha\beta} \cdot \hat{N}$$

and $$\hat{N} = \frac{X_1 \times X_2}{|X_1 \times X_2|}.$$

5(a). We can find the expression for principal curvature given by $K_n^2 - K_n \left(g^{\alpha\beta} d_{\alpha\beta}\right) + d/g = 0$ —— (1)

We know that a point is said to be umbilical point if the principal curvature be equal at that point. Hence for umbilical point roots of equation (1) will be equal i.e

$$\left(g^{\alpha\beta} d_{\alpha\beta}\right)^2 - 4\,d/g = 0$$

$$\Rightarrow \left(g^{\alpha\beta} d_{\alpha\beta}\right)^2 = 4\,d/g.$$

(b). Given that

$$\bar{r} = u e_1 + v e_2 + (u^2 + v^2) e_3$$

i.e. $$\bar{r} = (u, v, u^2 + v^2) \quad \text{——} \quad (1)$$

Using (1), we calculate $g^{\alpha\beta}$, $d_{\alpha\beta}$, ~~and~~ d and g at origin.

And these value will satisfy the condition

$$\left(g^{\alpha\beta} d_{\alpha\beta}\right)^2 = 4\, d/g$$

Hence origin will umbilical point of the surface.

6. We know that equation of involute is given by

$$X^i = x^i + (c-s)\, t^i \qquad \text{——— (I)}$$

differentiating above equation w.r.t. s_1, we have

$$T^i = \left[t^i + (c-s)\, \kappa p^i + t^i(-1)\right] \frac{ds}{ds_1}$$

$$T^i = (c-s)\, \kappa p^i \frac{ds}{ds_1} \qquad \text{——— (II)}$$

Taking ~~modulus we have~~

$$\Rightarrow \quad |T^i| = \left|(c-s)\, \kappa p^i \frac{ds}{ds_1}\right|$$

$$1 = (c-s)\, \kappa \frac{ds}{ds_1}$$

$$\Rightarrow \quad \frac{ds}{ds_1} = \frac{1}{(c-s)\kappa} \qquad \text{——— (III)}$$

Putting (III) in ~~(IV)~~ (II), we have

$$T^i = p^i$$

Again differentiating w.r.t. s_1, we have

$$\kappa_1 p^i = (\tau b^i - \kappa t^i) \frac{ds}{ds_1}$$

$$\Rightarrow \quad \kappa_1 p^i = \frac{(\tau b^i - \kappa t^i)}{\kappa(c-s)} \qquad \text{——— (IV)}$$

Differentiating (iv) w.r.t s_1, and using Serret-frenet formulae and putting ~~$\rho = 1/r_1$, σ~~ , we get

$$r_1 = \frac{\rho(\sigma\rho' - \sigma'\rho)}{(\rho^2+\sigma^2)(c-s)}.$$

7. Euler's Theorem: If α be the angle between a direction at a point P and the principal direction at P corresponding to K_1, then the normal curvature K_n in the direction is given by

$$K_n = K_1 \cos^2\alpha + K_2 \sin^2\alpha$$

Proof. We know that if the lines of curvature be parametric curves, then $g_{12} = 0 = d_{12}$. Then normal curvature is given by

$$K_n = d_{11}\left(\frac{du^1}{ds}\right)^2 + d_{22}\left(\frac{du^2}{ds}\right)^2 \quad \text{---(i)}$$

Since the principal curvature K_1 in the direction $du^2 = 0$, is given by

$$K_1 = \frac{d_{11}}{g_{11}} \quad \text{———— (ii)}$$

Similarly the principal curvature for the direction $du^1 = 0$, is given by $K_2 = \dfrac{d_{22}}{g_{22}}$ —— (iii)

We can ~~easily~~ show that

$$\left.\begin{aligned} \cos\alpha &= \sqrt{g_{11}}\; u'^1 \\ \text{and} \quad \cos\beta &= \sqrt{g_{22}}\; u'^2 \end{aligned}\right\} \text{———} (iv)$$

Since parametric curves are orthogonal $\beta = \frac{\pi}{2} - \alpha$, then (iv) become

$$\left.\begin{aligned} \cos\alpha &= \sqrt{g_{11}}\; u'^1 \\ \sin\alpha &= \sqrt{g_{22}}\; u'^2 \end{aligned}\right\} (v)$$

Putting values from (v) in (I), we have theorem.

<u>Q</u>- Given that

$$x^1 = u, \quad x^2 = v, \quad x^3 = u^2 + v^2$$

$$\Rightarrow \quad X = (u, v, u^2+v^2) \text{ ——— } (I)$$

The equation of lines of curvature is given by

$$\epsilon^{\delta r} g_{\delta\beta}\, d_{\alpha r}\, du^\alpha du^\beta = 0 \text{ ——— } (II)$$

~~we calcuate~~ Calculate all components of $g_{\delta\beta}$ and $d_{\alpha r}$ and putting in (II). we have a differential equation which solution will be given by $u^2 + v^2 = a^2$, $u = bv$, where a and b are constant.